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Dynamics of $N = 2$ supersymmetric Chern-Simons theories

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Abstract: We discuss several aspects of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories coupled to chiral multiplets. The generation of Chern-Simons couplings at low-energies results in novel behaviour including compact Coulomb branches and interesting patterns of dynamical supersymmetry breaking. We further show how, given any pair of mirror theories with $\mathcal{N} = 4$ supersymmetry, one may flow to a pair of mirror theories with $\mathcal{N} = 2$ supersymmetry by gauging a suitable combination of the R-symmetries. The resulting theories again have interesting properties due to Chern-Simons couplings.

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Contents

1. Introduction

Many three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry exhibit mirror symmetry, a phenomenon in which two theories with different ultraviolet descriptions flow to the same infrared physics. Initially discovered by Intriligator and Seiberg [1], many further pairs of mirror theories have since been constructed using various methods [2, 3]. Moreover, Kapustin and Strassler have suggested an elegant formula which captures many aspects of mirror symmetry in $\mathcal{N} = 4$ abelian gauge theories [4].

Under the mirror map, the two SU(2) R-symmetry groups of the three-dimensional theories are exchanged, together with the mass and Fayet-Iliopoulos (FI) parameters, and the Coulomb and Higgs branches. This latter exchange is particularly interesting as the Higgs branch metric is protected against quantum corrections, while the Coulomb branch metric may receive not only one-loop perturbative corrections, but also corrections from instantons, loops around the background of instantons and, more surprisingly, instanton-anti-instanton pairs. In the mirror theory, all of these effects are captured by a classical, hyperKähler quotient construction of the Higgs branch.

As with all dualities, it is natural to ask if one can continue to make progress with less supersymmetry. Indeed, several pairs of mirror theories with $\mathcal{N} = 2$ and, more speculatively, $\mathcal{N} = 1$ supersymmetry have been proposed [5]–[8]. In particular, the authors of [6] give a prescription for constructing such models given an $\mathcal{N} = 4$ pair. One first adds an $\mathcal{N} = 2$ multiplet to one theory, explicitly breaking half of the supersymmetry. If the scalar in this multiplet can be interpreted as dynamical mass parameter for some of the fields, then one may determine the mirror deformation: it must play the role of a dynamical FI parameter.

The first part of this paper will be concerned with providing an alternative prescription for flowing to $\mathcal{N}=2$ theories. The basic observation is very simple: there exists a combination of R-symmetries that may be gauged without fully breaking supersymmetry. Weakly gauging this symmetry introduces an axial mass, allowing a subset of the chiral multiplets to be integrated out. For abelian $\mathcal{N}=4$ theories, one finds that the resulting $\mathcal{N}=2$ theories have Chern-Simons couplings and several novel features, including the possibility of compact Coulomb branches. The abelian mirror pairs have been previously given in [7] where it was shown, using techniques of toric geometry, that the Higgs and Coulomb branches coincide. The method introduced here reproduces the results of [7], and may also be applied to theories with non-abelian gauge symmetry.

In section 3, we discuss the low-energy dynamics of $\mathcal{N}=2$ non-abelian theories with chiral multiplets. Perturbatively, the Coulomb branches exhibit similar behaviour to the abelian theories and, in particular, are compact. However instanton effects generate a superpotential which drives the vacuum to the boundary of the perturbative Coulomb branch. For $\mathrm{SU}(N)$ gauge groups with $N \geq 3$, this results in dynamical supersymmetry breaking. Section 4 summarises the main points.

2. Gauging the R-symmetry: abelian theories

We start by considering abelian theories. Later in this section, we will describe the most general abelian $\mathcal{N}=4$ mirror pairs, together with the deformation that breaks supersymmetry to $\mathcal{N}=2$. However, we first illustrate these ideas with a simple example which will also serve to set our notations and conventions.

The self-mirror theory. The simplest example of mirror symmetry in three dimensions is the $\mathcal{N}=4$ self-mirror theory of Intriligator and Seiberg [1]. It consists of a $\mathrm{U}(1)_G$ vector multiplet, together with two hypermultiplets. As we intend to partially break supersymmetry in the very near future, let us resort to four supercharge notation from the off. The $\mathcal{N}=4$ vector multiplet consists of an $\mathcal{N}=2$ vector multiplet V together with an $\mathcal{N}=2$ chiral multiplet Ψ. We denote the real scalar in the former as ϕ and the complex scalar in the latter as ψ. Together, these scalars transform in the $(\mathbf{3},\mathbf{1})$ of the $\mathrm{SU}(2)_N \times \mathrm{SU}(2)_R$ R-symmetry.

Each of the $i=1,2$ hypermultiplets consists of two chiral multiplets, Q_i and $\tilde{Q}_i$. These contain the complex scalars q_i and $\tilde{q}_i$ which have charge $+1$ and -1, respectively. These scalars transform in the $(\mathbf{1},\mathbf{2})$ representation of the R-symmetry.

The theory has two further global symmetries. Firstly there exists an $\mathrm{SU}(2)_F$ flavour symmetry which acts in the obvious fashion on the hypermultiplet index $i=1,2$. Weakly gauging this symmetry introduces a 3-vector of mass parameters, $\vec{m}$. These transform in the same manner as the vector multiplet scalars under the

R-symmetry and, without loss of generality, we may use this freedom to set $\vec{m} = (0, 0, m)$. This component of the mass vector is commonly referred to as a real mass; it is the complex mass that has been set to zero.

The second global symmetry is less obvious in the lagrangian formalism as it requires a dualisation to magnetic variables. Recall that in three dimensions one may exchange the photon in preference of a periodic scalar σ. Shifting this scalar by a constant, $\sigma \to \sigma + c$, is a symmetry of the theory at the classical and perturbative level. For abelian theories there are no instanton effects and this symmetry, which is usually denoted as $\mathrm{U}(1)_J$, survives in the full quantum theory. Once again, one may consider weakly gauging this symmetry[1] and the result is the introduction of a 3-vector of FI parameters $\vec{\zeta}$, transforming as $(\mathbf{1}, \mathbf{3})$ under the R-symmetry. The R-symmetry may again be employed to rotate the FI parameters to be $\vec{\zeta} = (0, 0, \zeta)$.

Notice in particular that in the presence of both mass and FI parameters, the R-symmetry is broken to its Cartan subalgebra, $\mathrm{U}(1)_N \times \mathrm{U}(1)_R$, while the flavour symmetry is similarly broken to $\mathrm{U}(1)_F$. It will prove useful in what follows to document the transformation of the hypermultiplet scalars under these various symmetries

	q_1	$\tilde{q}_1$	q_2	$\tilde{q}_2$
$\mathrm{U}(1)_G$	$+1$	-1	$+1$	-1
$\mathrm{U}(1)_F$	$+1$	-1	-1	$+1$
$\mathrm{U}(1)_R$	$+1$	$+1$	$+1$	$+1$

(2.1)

The scalar potential with both masses and FI parameters non zero is given by,

$$\begin{aligned} U = e^2\left(|q_i|^2 - |\tilde{q}_i|^2 - \zeta\right)^2 + e^2|\tilde{q}_i q_i|^2 + |\psi|^2\left(|q_i|^2 + |\tilde{q}_i|^2\right) + (\phi + m)^2|q_1|^2 + \\ + (-\phi - m)^2|\tilde{q}_1|^2 + (\phi - m)^2|q_2|^2 + (-\phi + m)^2|\tilde{q}_2|^2\,, \end{aligned} \tag{2.2}$$

where e is the gauge coupling constant which has dimension of $(\text{mass})^{1/2}$ and summation over i is assumed in the first three terms. The final four terms are the real masses for each of the chiral multiplets. The $\pm$ signs in front of ϕ and m reflect charges (2.1) of the scalars under $\mathrm{U}(1)_G$ and $\mathrm{U}(1)_F$, respectively. Importantly, $\mathcal{N} = 4$ supersymmetry ensures that $\mathrm{Mass}(q_i) = -\,\mathrm{Mass}(\tilde{q}_i)$.

The statement of mirror symmetry in this theory is that the Higgs branch and Coulomb branch metrics coincide [1]. Let us review how this comes about. For vanishing masses, the space of vacua is given by $\phi = \psi = 0$ while the hypermultiplet scalars are restricted by the first two terms in (2.2) modulo $\mathrm{U}(1)_G$. The result is that the theory possesses a four-dimensional Higgs branch with the Eguchi-Hanson metric given naturally by a hyperKähler quotient construction. The two-sphere which sits in the middle of this space has size ζ. This metric is classically exact: it receives no quantum corrections [9].

[1]This current couples to a twisted vector multiplet through a BF-coupling [4].

In contrast, when $\zeta = 0$, the space of vacua is given by $q_i = \tilde{q}_i = 0$, while ϕ, ψ and the dual photon σ are unconstrained, resulting in a four-dimensional Coulomb branch. While classically flat, the metric on this branch does receive quantum corrections. After integrating out the hypermultiplets at one loop, one finds the resulting metric is double-centered Taub-NUT space which, in the limit $e^2 \to \infty$ becomes once again Eguchi-Hanson. The two-sphere in the center has size m. There are no higher loop corrections to this metric.

Thus this theory is self-mirror [1]: in the infrared limit $e^2 \to \infty$, the low-energy physics on the Coulomb branch is the same as that on the Higgs branch *if* one swaps mass and FI parameters and $\mathrm{SU}(2)_N$ and $\mathrm{SU}(2)_R$ symmetries.

It is instructive to compare the isometries of the two spaces. These descend from the symmetries of the theories. The Higgs branch has an $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$ symmetry, reflecting the transformation of the hypermultiplet scalars discussed earlier. Meanwhile, for finite e^2, the Coulomb branch has a $\mathrm{U}(1)_J \times \mathrm{U}(1)_N$ isometry. However, in the strong coupling this is enhanced to $\mathrm{SU}(2)_J \times \mathrm{U}(1)_N$. Such enhancement of global flavour symmetries in the infrared is commonplace on the Coulomb branch of mirror pairs and is still rather poorly understood.

Gauging the R-symmetry. Having reviewed mirror symmetry in this self-mirror case, let us proceed to break supersymmetry to $\mathcal{N} = 2$. The basic idea is simple: just as we weakly gauged the global flavour and shift symmetries above in order to introduce background parameters, we may perform a similar operation with the R-symmetry. Of course, usually gauging the R-symmetry results in supersymmetry being completely broken. In order to see this, we need only observe that the superpotential of a theory transforms under the R-symmetry and therefore it is not possible to write the lagrangian in a manifestly gauge invariant and supersymmetric fashion. In the present case, the superpotential is given by

$$\mathcal{W} = \tilde{Q}_i \Psi Q_i \,.$$

For non-zero mass and FI parameters, the residual R-symmetry group is $\mathrm{U}(1)_N \times \mathrm{U}(1)_R$, under which Q_i and $\tilde{Q}_i$ both have charge $(0, +1)$ while Ψ has charge $(+2, 0)$. We see that the superpotential indeed transforms under each of the R-symmetries. However, it is clear that if we choose to gauge only the axial combination, $\mathrm{U}(1)_{R-N}$, then the superpotential is invariant and $\mathcal{N} = 2$ supersymmetry will remain manifestly unbroken. Moreover, as mirror symmetry exchanges the two R-symmetries, the mirror deformation must require the same current to be gauged, albeit with the introduction of a minus sign: $\mathrm{U}(1)_{N-R}$. For the remainder of this section, we will bring this simple trick to bear on various theories. For now, let us content ourselves with examining the consequences in the simple case under discussion.

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In weakly gauging $\mathrm{U}(1)_{R-N}$, we introduce a single, real[2] background parameter X. From the charges assigned to the various fields, one may determine the new scalar potential,

$$
\begin{aligned}
U = & \, e^2\left(|q_i|^2 - |\tilde{q}_i|^2 - \zeta\right)^2 + e^2|\tilde{q}_i q_i|^2 + |\psi|^2\left(|q_i|^2 + |\tilde{q}_i|^2\right) + 4X^2|\psi|^2 + \\
& + (\phi + m \pm X)^2|q_1|^2 + (-\phi - m \pm X)^2|\tilde{q}_1|^2 + \\
& + (\phi - m \pm X)^2|q_2|^2 + (-\phi + m \pm X)^2|\tilde{q}_2|^2 \,, \qquad (2.3)
\end{aligned}
$$

where the $\pm$ signs in front of the X's depend on whether we choose to gauge $\mathrm{U}(1)_{R-N}$ (plus sign by convention) or $\mathrm{U}(1)_{N-R}$ (minus sign). We will refer to the case with $+X$ as theory A, and the case with $-X$ as theory B.

The effect of gauging the R-symmetry is two-fold: the chiral multiplet Ψ has gained a mass, and a mass splitting has been introduced between Q_i and $\tilde{Q}_i$. Although we have exhibited mass terms only for scalars, $\mathcal{N}=2$ supersymmetry ensures that the fermionic superpartners have a similar splitting. The claim is that, by construction, the theory with deformation $+X$ is mirror to the theory with deformation $-X$. Let us now demonstrate this explicitly in the limit that $X \to \infty$. In doing so, we will uncover several interesting properties of these theories.

Firstly, consider theory A. Clearly we do not want to integrate out all fields, so we perform the shift,

$$\phi' = \phi + X$$

and require that m, together with the rescaled scalar ϕ', remain finite as $X \to \infty$. The chiral multiplets Ψ, $\tilde{Q}_1$ and $\tilde{Q}_2$ thus become heavy and may be integrated out. The Ψ field is uncharged under the gauge field and simply decouples as its mass becomes infinite.

However, things are somewhat more interesting for the charged chirals, for even in the limit of infinite mass they do not decouple completely. In particular, when integrated out, they generate a Chern-Simons coupling [11, 12],

$$\frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}$$

together with its supersymmetric completion. Importantly, as will be discussed shortly, this supersymmetric completion includes a contribution to the scalar potential [13]. Consider the more general situation in which we have N chiral superfields of charge s_i and real mass M_i. Integrating out these fields induces a Chern-Simons coupling

$$\kappa = \frac{1}{2}\sum_{i=1}^{N} s_i^2 \, \mathrm{sign}(M_i)\,. \qquad (2.4)$$

[2]Previously we gauged the background symmetries in an $\mathcal{N}=4$ supersymmetric fashion resulting in three background parameters, reflecting the existence of three real scalars in the $\mathcal{N}=4$ vector multiplet. In the current situation, we wish to preserve $\mathcal{N}=2$ supersymmetry, and thus introduce only a single real background parameter.

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A very nice review of many properties of Chern-Simons theories, including a discussion of these dynamically generated terms, can be found in [14]. Equation (2.4) shows us the importance of the previous observation that $\mathrm{Mass}(q_i) = -\mathrm{Mass}(\tilde{q}_i)$. This ensures that in $\mathcal{N} = 4$ supersymmetric theories, we may integrate out hypermultiplets with impunity without fear of generating CS terms. The same is not true of chiral multiplets in $\mathcal{N} = 2$ theories.

Returning to the example in hand, the dynamically generated CS parameter is given by,

$$\kappa = \frac{1}{2}\Big(\,\mathrm{sign}(-\phi' - m + 2X) + \mathrm{sign}(-\phi' + m + 2X)\Big) = +1\,.$$

At the same time, there is also a finite renormalisation of the FI parameter. The simplest way to see this is to notice that the FI parameter plays the role of a cross CS term between $\mathrm{U}(1)_G$ and both $\mathrm{U}(1)_F$ and $\mathrm{U}(1)_{R-N}$. While the former does not contribute, the latter gives,

$$\zeta \longrightarrow \zeta' = \zeta - X\,. \tag{2.5}$$

And, naturally, we take this rescaled FI parameter to be finite. Thus we are left with:

Theory A: $\mathcal{N} = 2$ U(1) gauge theory with two chiral multiplets, both of charge $+1$, with real masses $+m$ and $-m$, respectively, a CS parameter $+1$ and FI parameter ζ'.

It will be instructive to examine the scalar potential of this theory,

$$U_A = e^2\Big(|q_1|^2 + |q_2|^2 - \kappa\phi - \zeta'\Big)^2 + (\phi + m)^2|q_1|^2 + (\phi - m)^2|q_2|^2\,, \tag{2.6}$$

with $\kappa = +1$. Notice in particular the presence of the term $\kappa\phi$ inside the D-term. This is part of the supersymmetric completion of the CS term mentioned previously.

We turn now to the mirror theory. Noting that theory A required a shift of the FI parameter (2.5), it is natural to conjecture that theory B requires a similar shift of the mass parameter,

$$m' = m - X\,.$$

Now in the limit $X \to \infty$, with ϕ and m' kept fixed, we are forced to integrate out ψ, $\tilde{q}_1$ and q_2. Notice that this is a different combination of chiral fields from the previous case, which means that while a CS parameter is still generated, $\kappa = -1$, the FI parameter is not renormalised in this case. Thus we are led to

Theory B: $\mathcal{N} = 2$ U(1) gauge theory with two chiral multiplets, of charge $+1$ and -1, both with real masses $+m'$, a CS parameter -1 and FI parameter ζ.

The scalar potential for this theory is given by

$$U_B = e^2\Big(|q_1|^2 - |\tilde{q}_2|^2 - \kappa\phi - \zeta\Big)^2 + (\phi + m')^2|q_1|^2 + (-\phi + m')^2|\tilde{q}_2|^2\,, \tag{2.7}$$

with $\kappa = -1$.

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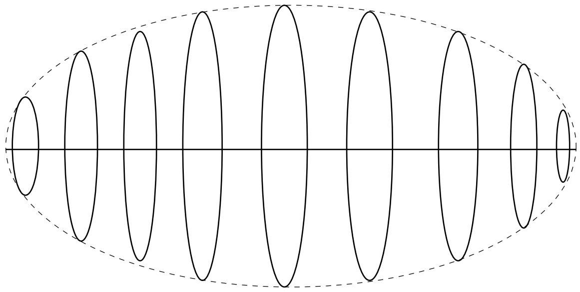

Figure 1: The Coulomb branch of theory B. The scalar ϕ (plotted horizontally) is restricted to lie within the interval $-m < \phi < m$. The dual photon provides a further $\mathbf{S}^1$ which is fibered over this interval such that its radius vanishes at the end points, resulting in a Coulomb branch with topology of the sphere.

It was shown in [7] that the vacuum moduli spaces of theories A and B do indeed coincide if we exchange m with ζ and m' with ζ'. Here we review the basic features of mirror symmetry in these theories. Firstly, we consider the Higgs branch of theory A. This exists if $m = 0$ and is well known to be a copy of $\mathbb{CP}^1$ of Kähler class $\zeta' > 0$. The challenge is to reproduce this as the Coulomb branch of theory B when $\zeta = 0$.

The observation of [7] is that while classically the presence of the CS parameter in (2.7) ensures that there is no Coulomb branch, this situation is improved by quantum effects. Specifically, let us set $q_1 = \tilde{q}_2 = 0$, and $\phi \neq 0$. Then, upon integrating out the remaining chiral multiplets, the CS parameter receives a further correction,

$$\kappa \longrightarrow -1 + \frac{1}{2}\Big(\,\mathrm{sign}(\phi + m') + \mathrm{sign}(-\phi + m')\Big)\,.$$

The existence of the Coulomb branch hinges on the vanishing of this quantity, which occurs only when $|\phi| \leq m' > 0$. Thus the dynamical generation of CS terms restricts the Coulomb branch to the interval $-m' \leq \phi \leq m'$.

This is the first part of the story. The second part concerns the dual photon σ, which provides us with a circle fibered over the interval. It was argued in [6, 7] that the end points of the interval must be fixed points of the $\mathrm{U}(1)_J$ symmetry as there exist extra massless degrees of freedom at these points which are invariant under $\mathrm{U}(1)_J$. The circle therefore shrinks to zero size at the end points of the interval. The resulting Coulomb branch is depicted in figure 1.

Classically the metric on the Coulomb branch is given by

$$ds^2 = H d\phi^2 + H^{-1} d\sigma^2\,,$$

where $H = 1/e^2$. While the Kähler potentials of theories with four supercharges are generically not protected against quantum corrections, in this theory we may argue that the metric receives only one-loop corrections. To see this recall that mirror symmetry requires the Coulomb branch to develop an enhanced $\mathrm{SU}(2)_J$ isometry group in the infrared which, given the topology, restricts the metric to be Fubini-Study. Indeed, at one-loop, we find

$$H_{\text{one-loop}} = \frac{1}{e^2} + \frac{1/2}{|\phi + m|} + \frac{1/2}{|-\phi + m|} = \frac{1}{e^2} + \frac{m}{\phi^2 - m^2}$$

which becomes Fubini-Study as $e^2 \to \infty$.

General abelian theories. The most general $\mathcal{N} = 4$ abelian mirror pairs were exhibited in [3]. The following presentation differs somewhat from that reference, but may easily be seen to be equivalent. In particular, the following formalism of mirror symmetry arises naturally from the Kapustin-Strassler formula [4]. The mirror pairs are:

Theory A: $\mathcal{N} = 4$ $\mathrm{U}(1)^r$ gauge theory with N hypermultiplets of charge R_i^a, $i = 1, \ldots, N$ and $a = 1, \ldots, r$, where $N \geq r$ and the charge matrix is taken to have maximal rank. The theory has FI parameters $\vec{\zeta}^a$ and mass parameters $\vec{m}_i$, each of which is a 3-vector. Notice that only $N - r$ of the mass parameters are independent.

Theory B: $\mathcal{N} = 4$ $\mathrm{U}(1)^{N-r}$ gauge theory with N hypermultiplets of charge S_i^p, $i = 1, \ldots, N$ and $p = 1, \ldots, N - r$. Again, the charge matrix is taken to be of maximal rank. This theory has vector FI parameters $\hat{\vec{\zeta}}^p$ and mass parameters $\hat{\vec{m}}_i$, where $N - (N - r) = r$ of the mass parameters are independent.

The charges of theories A and B are constrained to satisfy

$$\sum_{i=1}^{N} R_i^a S_i^p = 0 \quad \text{for all } a \text{ and } p\,. \tag{2.8}$$

This restriction ensures that the charges S_i^p may be thought of as a basis of generators for the Cartan subalgebra of the flavour symmetry of theory A. Likewise, R_i^a provide a basis for the abelian flavour symmetry of theory B. The relationship between the mass parameters of theory A and the FI parameters of theory B is most conveniently described in terms of N mirror invariant 3-vectors, $(\vec{n}_a, \hat{\vec{n}}_p)$, through

$$\begin{aligned} \vec{\zeta}^a &= R_i^a R_i^b \vec{n}_b\,, & \vec{m}_i &= S_i^p \hat{\vec{n}}_p\,, \\ \hat{\vec{\zeta}}^p &= S_i^p S_i^q \hat{\vec{n}}_q\,, & \hat{\vec{m}}_i &= R_i^a \vec{n}_a \end{aligned} \tag{2.9}$$

from which we find the desired relations $\vec{\zeta}^a = R_i^a \hat{\vec{m}}_i$ and $\hat{\vec{\zeta}}^p = S_i^p \vec{m}_i$.

The statement of mirror symmetry is that the classical metric on the Higgs branch of theory A coincides with the one-loop corrected metric on the Coulomb branch of theory B in the infrared limit $e^2 \to \infty$ and vice-versa, if the masses and FI parameters are related as above.

For generic values of the mirror-invariants, the $\mathrm{SU}(2)_N \times \mathrm{SU}(2)_R$ R-symmetry of the theory is completely broken. This will not do for our purposes. We therefore restrict attention to the subset of parameter space in which $\vec{n}_a = (0,0,n_a)$ and $\hat{\vec{n}}_q = (0,0,\hat{n}_q)$, such that all complex mass and FI parameters are set to zero. This ensures that the residual R-symmetry is once again $\mathrm{U}(1)_N \times \mathrm{U}(1)_R$.

Weakly gauging the axial combination $\mathrm{U}(1)_{R-N}$ with a background parameter X again introduces a mass for each chiral multiplet in the vector multiplets of theory A. As previously, these simply decouple in the limit $X \to \infty$ and the interesting physics occurs due to the mass splitting of the hypermultiplets. Specifically, the masses of the hypermultiplets of theory A are given by

$$\sum_{i=1}^{N} \left(R_i^a \phi_a + m_i + X\right)^2 |q_i|^2 + \sum_{i=1}^{N} \left(-R_i^a \phi_a - m_i + X\right)^2 |\tilde{q}_i|^2 . \tag{2.10}$$

Once again, we must perform a rescaling of fields before integrating out those we deem to be heavy. We choose the rescaling,

$$R_i^a \phi_a' = R_i^a \phi_a + \frac{1}{2} X , \qquad m_i' = m_i + \frac{1}{2} X \tag{2.11}$$

in terms of which, the masses (2.10) are given by

$$\sum_{i=1}^{N} \left(R_i^a \phi_a' + m_i'\right)^2 |q_i|^2 + \sum_{i=1}^{N} \left(-R_i^a \phi_a' - m_i' + 2X\right)^2 |\tilde{q}_i|^2 \tag{2.12}$$

leaving us with the task of decoupling the $\tilde{q}_i$'s. The induced CS couplings now include the possibility of cross-terms of the form $(1/4\pi)\kappa^{ab}\epsilon_{\mu\nu\rho}A_a^\mu F_b^{\nu\rho}$ and the generalisation of (2.4) to include such cross terms is given by,

$$\kappa^{ab} = \frac{1}{2} \sum_{i=1}^{N} R_i^a R_i^b \, \mathrm{sign}(M_i) = \frac{1}{2} R_i^a R_i^b , \tag{2.13}$$

where M_i is the mass of the i-th chiral multiplet which, in the present case, is necessarily positive for $\tilde{q}_i$ in the limit $X \to \infty$. Similarly, there is a finite renormalisation of the FI parameter,

$$\zeta^a \longrightarrow \zeta'^a = \zeta_a - \frac{1}{2} \sum_{i=1}^{N} (X - m_i) R_i^a = \zeta_a - \frac{1}{2} X \sum_{i=1}^{N} R_i^a , \tag{2.14}$$

where the equality requires the use of the relationships (2.8) and (2.9). Finally, we are left with,

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Theory A: $\mathcal{N} = 2$ $U(1)^r$ gauge theory with N chiral multiplets of charge R_i^a, $i = 1, \ldots, N$ and $a = 1, \ldots, r$ with CS parameters $\kappa^{ab} = \frac{1}{2} R_i^a R_i^b$, FI parameters ζ'^a, and mass parameters m_i'.

The prescription to compute the mirror theory is clear. Firstly one should perform the mirror rescaling of the parameters. For the masses the mirror of (2.14), is simply $\hat{m}_i' = \hat{m}_i - \frac{1}{2}X$. However, the rescaling of the FI parameters is a quantum effect, arising from integrating out the chiral multiplets. The correct rescaling is induced by a shift of the vector multiplet scalars, $\hat{\phi}_p' = \hat{\phi}_p - \frac{1}{2}X$. After weakly gauging $U(1)_{N-R}$, the masses of the hypermultiplets in theory B are given in terms of these rescaled parameters as

$$\sum_{i=1}^{N} \left(S_i^p \hat{\phi}_p' + \hat{m}_i \right)^2 |q_i|^2 + \sum_{i=1}^{N} \left(-S_i^p \hat{\phi}_p' - \hat{m}_i - 2X \right)^2 |\tilde{q}_i|^2 . \tag{2.15}$$

Integrating out the $\tilde{q}_i$ as $X \to \infty$ induces the CS parameters $\hat{\kappa}^{pq} = -\frac{1}{2} S_i^p S_i^q$, together with the finite renormalisation of the FI parameters, $\hat{\zeta}^p \to \hat{\zeta}'^p = \zeta^p + \frac{1}{2} X \sum_{i=1}^{N} S_i^p$, which is indeed the mirror deformation of (2.11). Thus the final result is,

Theory B: $\mathcal{N} = 2$ $U(1)^{N-r}$ gauge theory with N chiral multiplets of charge S_i^p, $i = 1, \ldots, N$ and $p = 1, \ldots, N-r$, with CS parameters $\hat{\kappa}^{pq} = -\frac{1}{2} S_i^p S_i^q$, FI parameters $\hat{\zeta}'^p$ and mass parameters $\hat{m}_i'$.

This theory is mirror to theory A when the masses and FI parameters are related through the relevant mirror invariant quantities (2.9). It was shown in [7] that the Coulomb branch of theory A is equivalent as a toric variety to the Higgs branch of theory B and vice versa. The latter is an r-dimensional complex space given by the usual symplectic quotient construction. The Coulomb branch is more interesting: the requirement that the effective CS parameter vanishes after integrating out the remaining chiral multiplets restricts ϕ variables to lie within a region $\Delta \subset \mathbb{R}^r$. The dual photons provide a torus $\mathbb{T}^r$ which is fibered over Δ such that certain cycles shrink at the boundaries. The resulting space is equivalent to the Higgs branch.

An example: $\mathbb{CP}^2$. Let us illustrate the construction of the Coulomb branch as a toric variety with a particularly simple example. For further details, including a description of the most general case, see [7]. It is well known how to construct $\mathbb{CP}^2$ as the Higgs branch of theory A: we must take a single $U(1)$ gauge factor with 3 chiral multiplets, each of charge $+1$. Setting the masses to zero ensures that the vacuum moduli space is given by the vanishing of the D-term,

$$\sum_{i=1}^{3} |q_i|^2 = \zeta$$

which, after modding out by gauge transformations $q_i \to e^{i\alpha} q_i$, leads to the promised projective space.

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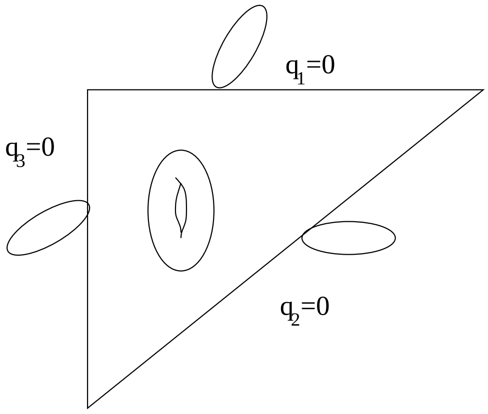

Figure 2: The toric realisation of $\mathbb{CP}^2$. A torus $\mathbb{T}^2$ is fibered over the triangle such that a single cycle degenerates at each edge. Each of these edges is itself a copy of $\mathbb{CP}^1$ of the form shown in figure 1.

Before turning to theory B, let us first describe $\mathbb{CP}^2$ in more detail, following the discussion of [15]. At a generic point, $\mathbb{CP}^2$ admits an action of $\mathbb{T}^2$, parametrised by the periodic coordinates θ_1 and θ_2. We may choose a basis of this to be

$$(q_1, q_2, q_3) \longrightarrow \left(e^{i\theta_1} q_1, e^{i(\theta_2-\theta_1)} q_2, e^{-i\theta_2} q_3\right).$$

It is interesting to look at the fixed points of this action. There are three combinations of cycles which have fixed points: the cycle $\theta_1 + 2\theta_2$ degenerates on the $\mathbb{CP}^1$ submanifold $q_1 = 0$; the cycle $\theta_2 + 2\theta_1$ degenerates on the $\mathbb{CP}^1$ submanifold $q_3 = 0$; the cycle $\theta_1 - \theta_2$ degenerates on the $\mathbb{CP}^1$ submanifold $q_2 = 0$. There are also three fixed points where two out of three of these cycles vanish. These observations allow us to represent $\mathbb{CP}^2$ as shown in figure 2.

Let us now turn to the Coulomb branch of theory B which, in this case is $\mathrm{U}(1)^2$ with 3 chiral multiplets of charge S_i^p given by $(1,0)$, $(-1,1)$ and $(0,-1)$. Each has mass $m = \frac{2}{3}\zeta$. As described above, the bare CS coupling is given by $\kappa^{pq} = -\frac{1}{2} S_i^p S_i^q$, but on the Coulomb branch, it also receives contributions from the remaining chiral multiplets, so that the effective CS parameter is given by,

$$\kappa^{pq} = \frac{1}{2} \sum_{i=1}^{3} S_i^p S_i^q \left(-1 + \mathrm{sign}(M_i)\right),$$

where M_i is the effective mass of the i-th chiral multiplet, given by

$$M_1 = \hat{\phi}_1 + m, \qquad M_2 = -\hat{\phi}_1 + \hat{\phi}_2 + m, \qquad M_3 = -\hat{\phi}_2 + m.$$

The Coulomb branch exists for $\kappa^{pq} = 0$, which requires $\mathrm{sign}(M_i) = +1$ for $i = 1, 2, 3$, and therefore restricts the Coulomb branch to the region shown in figure 3.

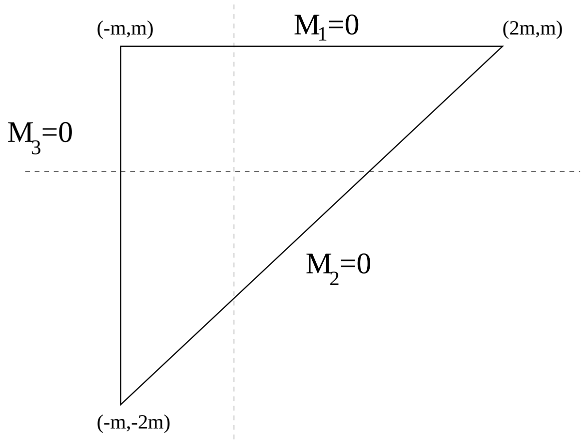

Figure 3: The Coulomb branch of theory B. Outside the triangle, CS terms develop, lifting the Coulomb branch. The two dual photons provide a torus $\mathbb{T}^2$ which is fibered over the triangle to realise $\mathbb{CP}^2$ as shown in figure 2.

The remaining part of the Coulomb branch arises from the two dual photons, σ_1 and σ_2 which supply a two torus $\mathbb{T}^2$ at generic points of the Coulomb branch. However, at the boundary of the Coulomb branch, the chiral multiplet M_i becomes massless and one of the cycles of the $\mathbb{T}^2$ shrinks. In order to see which cycle is shrinking, note that when $M_i = 0$, a Higgs branch emerges from the boundary if we gauge the $\mathrm{U}(1)_J$ current that induces a constant shift of the combination $S_i^a \sigma_a$ [7]. This therefore is the shrinking cycle. Relating the phases that we introduced on the Higgs branch with the dual photons via

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

completes the identification of Coulomb and Higgs branches.

Magnetic coupling. The mirror map exchanging mass with FI parameters relates the global symmetries of the two theories. In particular, weakly gauging the global flavour symmetry of theory A introduces mass parameters. The mirror deformation, namely introducing FI parameters, can be achieved by weakly gauging the global $\mathrm{U}(1)_J$ symmetries of theory B.

In [4], Kapustin and Strassler pointed out that one need not be so weak when gauging these symmetries. In other words, the newly introduced coupling constants may be kept finite and mirror symmetry still holds. The gauge potential of the flavour symmetries couples to the hypermultiplets in the standard, electric, fashion, with coupling constant e. However, the $\mathrm{U}(1)_J$ symmetries couple to the hypermultiplets through a BF-coupling (dubbed four-fermi coupling in [4]). The resulting coupling constant, $\tilde{e}$, is often referred to as the "magnetic coupling". Although

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this terminology is somewhat misleading, it is widespread in the literature so we retain it here with the caveat that this is not the electric-magnetic dual coupling — see [4].

Of course, we know only the relationship between the Cartan subalgebras of the global symmetries: the full non-abelian symmetry of the Coulomb branch is not seen in the classical lagrangian. This limits our identification of this magnetic coupling to abelian gauge theories. Presumably, a better understanding of enhanced global symmetries in these theories may suggest a way to extend this technique to non-abelian gauge theories as well.

In this subsection, we merely comment that the operations described above are compatible with the partial breaking of supersymmetry. For example, we may consider the simple $\mathcal{N}=4$ model of U(1) with a single hypermultiplet and finite coupling constant e. Weakly gauging the $\mathrm{U}(1)_{R-N}$ symmetry of this model as described above leads to $\mathcal{N}=2$ U(1) gauge theory with a single chiral multiplet of charge $+1$ and a CS parameter $\kappa=+1/2$. The resulting Coulomb branch exists only for $\phi\leq 0$ and has one-loop metric

$$ds^2=\left(\frac{1}{e^2}+\frac{1}{2|\phi|}\right)d\phi^2+\left(\frac{1}{e^2}+\frac{1}{2|\phi|}\right)^{-1}d\sigma^2 \quad (\phi\leq 0)\,. \tag{2.16}$$

On the other side, the mirror $\mathcal{N}=4$ theory is given by $\mathrm{U}(1)_1\times\mathrm{U}(1)_2$ gauge group with a single hypermultiplet which is coupled only to the first U(1) factor. While the gauge coupling e_1 is sent to infinity, e_2 may be kept finite and is to be identified with the coupling e of theory A under mirror symmetry. The two vector multiplets are coupled through a BF term.[3] Weakly gauging $\mathrm{U}(1)_{N-R}$ of this theory results in the $\mathcal{N}=2$ theory with $\mathrm{U}(1)^2$ gauge group and a single chiral multiplet of charge $(+1,0)$. The bare CS coupling is given by

$$\kappa^{ab}=\begin{pmatrix}-\frac{1}{2} & +\frac{1}{2}\\ +\frac{1}{2} & 0\end{pmatrix},$$

where the off-diagonal terms are the $\mathcal{N}=2$ relic of the $\mathcal{N}=4$ BF-coupling and κ^{11} is induced from integrating out the chiral multiplet. The scalar potential of this theory is given by,

$$U=e_1^2\left(|q|^2-\frac{1}{2}\hat{\phi}_1-\hat{\phi}_2\right)^2+e_2^2(\hat{\phi}_1)^2+\hat{\phi}_1^2|q|^2$$

which has a Higgs branch given by $\hat{\phi}_1=0$, with q and $\hat{\phi}_2$ constrained to satisfy the first D-term, modulo $\mathrm{U}(1)_1$ gauge transformations. Importantly, the dual photon $\hat{\sigma}_2$ transforms as $\hat{\sigma}_2\to\hat{\sigma}_2+\alpha$ under such transformations [4]. It may be checked explicitly that the resulting Kähler quotient reproduces the metric (2.16) above.

[3]Strictly speaking $\mathrm{U}(1)_2$ is a twisted vector multiplet. This distinction no longer exists in $\mathcal{N}=2$ theories. See [4] for further details.

Finally note that, in the spirit of [4], we may also consider keeping a finite coupling constant for the gauged $\mathrm{U}(1)_{R-N}$ symmetry, preserving the property of mirror symmetry.

3. Non-abelian Chern-Simons theories

The idea of gauging the R-symmetry to flow from $\mathcal{N}=4$ to $\mathcal{N}=2$ theories applies equally well to non-abelian mirror pairs, of which many are known. Here we will discuss the dynamics of the class of theories that arise in such a construction.

Let us recall a few relevant facts concerning the non-abelian CS coupling,

$$\mathcal{L}_{\mathrm{CS}} = \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu[A_\nu, A_\rho]\right). \tag{3.1}$$

Unlike in the abelian case, the action is not gauge invariant. The requirement of invariance of the partition function under gauge transformations that that are homotopically non trivial in $\pi_3(G)$ requires the CS parameter to be quantised: $\kappa \in \mathbb{Z}$. For non-abelian groups, gauge invariance does not allow for the possibility of cross CS terms of the form (2.13).

In non-supersymmetric pure Yang-Mills Chern-Simons theories, the CS parameter κ undergoes a finite, integer renormalisation [16]. This is not the case in the $\mathcal{N}=2$ theories considered here [17]. However, as before, κ is renormalised by integrating out massive chiral multiplets. Specifically, consider N chiral multiplets of real mass M_i, transforming in the representation R_i of some unbroken semi-simple gauge group G. Integrating out this matter results in the effective CS parameter

$$\kappa \longrightarrow \kappa + \frac{1}{2}\sum_{i=1}^{N} d_2(R_i)\,\mathrm{sign}(M_i)\,, \tag{3.2}$$

where $d_2(R_i)$ is the quadratic casimir of R_i normalised such that $d_2(\mathbf{N}) = 1$ for the fundamental representation of $\mathrm{SU}(N)$. Notice in particular, the factor of $1/2$ in (3.2) implies that an $\mathcal{N}=2$ $\mathrm{SU}(N)$ theory with a single chiral multiplet in the fundamental representation is consistent only if one adds a bare half-integer CS parameter. Such a term breaks parity at the classical level and is referred to as a parity anomaly [11, 12]. As in the abelian case, it is possible for parity to be restored in the effective theory after integrating out the chiral multiplets.

To illustrate mirror symmetry, let us consider the simplest non-abelian mirror pairs discovered by Intriligator and Seiberg [1]

Theory A: $\mathcal{N}=4$ $\mathrm{SU}(2)$ gauge theory with $N \geq 4$ hypermultiplets in the **2** representation.

Theory B: $\mathcal{N} = 4$ D_N quiver theory with gauge group

$$G_{D_N} = \prod_{a=1}^{N+1} \frac{U(n_a)}{\mathrm{U}(1)},$$

where $n_a = 1$ for $a = 1, 2, 3, 4$ and $n_a = 2$ for $a = 5, \ldots, N+1$ are the Dynkin indices of the D_N diagram. Bi-fundamental hypermultiplets transform in the $c_{ab}(\mathbf{n}_a, \bar{\mathbf{n}}_b)$ where $c_{ab} = 1$ if a link joins the a-th and b-th node, and is zero otherwise — see [1] for further details.

Gauging the $\mathrm{U}(1)_{R-N}$ symmetry to introduce an axial mass proceeds in the same manner as in the previous section. The only difference is that the adjoint chiral superfield Ψ now also contributes to the CS parameter. For SU(2) gauge group, the casimirs are related by $d_2(\mathbf{3}) = 4d_2(\mathbf{2})$. After a suitable shift of the mass parameters, theory A flows to

Theory A: $\mathcal{N} = 2$ SU(2) gauge theory with N chiral multiplets in the $\mathbf{2}$ representation with non-abelian CS parameter $\kappa = \frac{1}{2}N - 2$.

For the quiver theory, each $\mathrm{SU}(2) \subset G_{D_N}$ has four fundamental hypermultiplets. After a suitable shift of the U(1) scalars, the fundamental and adjoint chiral multiplet cancel in their contribution to the non-abelian CS parameter. The abelian CS parameters do receive contributions however. We find,

Theory B: $\mathcal{N} = 2$ D_N quiver theory, with gauge group G_{D_N} with bi-fundamental chiral multiplets determined by the connections of the D_N Dynkin diagram. The non-abelian CS parameter is zero, while the abelian CS parameters are given by $\kappa_{aa} = -1$ for $a = 1, 2, 3, 4$ and $\kappa_{aa} = -4$ for $a = 5, \ldots, N+1$ and $\kappa_{ab} = +\frac{1}{2}c_{ab}n_a n_b$ if $a \neq b$.

Rather than enter into the details of elucidating agreement between the vacuum structure of these two theories, we will instead concern ourselves with a discussion of the class of theories that arise, namely non-abelian gauge theories with chiral multiplets.

SU(2) **theories.** We start by discussing the simplest non-abelian theory: $\mathcal{N} = 2$ SU(2) gauge theory with zero bare CS parameter. The vector multiplet includes a real scalar ϕ in the adjoint representation. We further include two chiral multiplets in the $\mathbf{2}$ representation with real masses $-m$ and $+M$ where we choose $m, M > 0$.

Integrating out the massive chirals will result in CS couplings. Let us first position ourselves far out on the Coulomb branch, breaking $\mathrm{SU}(2) \to \mathrm{U}(1)$ with $\phi = v\tau^3$, where τ^3 is the third Pauli matrix. The effective description is in terms of the surviving U(1) gauge group. Thus the non-abelian CS couplings (3.1) are not relevant in determining the vacuum structure and we must look once more to the now-familiar abelian CS coupling.

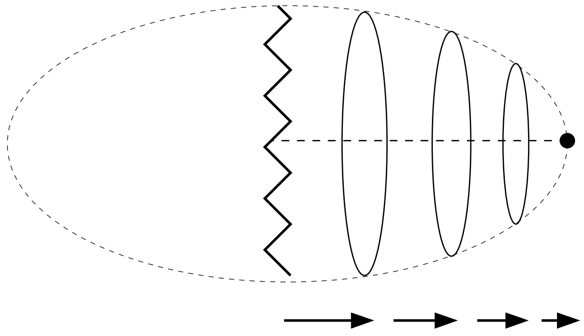

Figure 4: The Coulomb branch of the SU(2) theory in the limit $M \to \infty$. Non-abelian gauge symmetry is restored at the origin, denoted by the vertical jagged line. Instantons, represented by the arrows, induce a potential driving the vacuum to the point marked by the solid dot. This theory has a unique supersymmetric vacuum.

Each chiral multiplet, q_i (with $i = 1, 2$ a flavour index) decomposes into two chirals q_i^a under the unbroken U(1) (with $a = 1, 2$ a colour index). The chirals q_i^1 have charge $+1$ while q_i^2 have charge -1. Their masses are given by,

$$(v-m)^2 \left|q_1^1\right|^2 + (-v-m)^2 \left|q_1^2\right|^2 + (v+M)^2 \left|q_2^1\right|^2 + (-v+M)^2 \left|q_2^2\right|^2 .$$

Notice that if $M = m$ then the chiral multiplets come in pairs with opposite real masses. In such a case, no CS parameter will be generated. We will therefore restrict attention to a somewhat different limit: $M \to \infty$ with v and m kept finite. Integrating out q_2^a then generates an abelian CS coupling with $\kappa = \frac{1}{2} \times 2 \times \mathrm{sign}(M) = +1$. We must also integrate out the W-boson multiplets. However the these come with opposite charges and opposite real masses induced by the Higgs mechanism and so do not contribute to κ. Following the discussion in the previous section, we see that the Coulomb branch exists if, after integrating out the remaining chiral multiplets q_1^a, the effective CS parameter vanishes,

$$\kappa = +1 + \frac{1}{2}\mathrm{sign}(v-m) + \frac{1}{2}\mathrm{sign}(-v-m) = 0$$

which requires $|v| < m$. This is the same restriction we found in the U(1) theory with two chirals. This, of course, is not surprising as, after the Higgs mechanism, the matter content coincided with the abelian model. However, the dynamics of this theory remember their non-abelian origin, and therefore differ from the U(1) theory, in three distinct ways. Firstly, the residual Weyl symmetry allows us to fix $0 < v < m$. Secondly, the description of the Coulomb branch in terms of a periodic scalar fibered over the interval breaks down at $v = 0$ where non-abelian gauge symmetry is classically restored and we cannot perform the manoeuvres necessary to dualise the gauge field. The resulting perturbative Coulomb branch is shown in figure 4.

In fact, in this model the troublesome point at $v = 0$ is actually removed from consideration by the third difference: instantons. In three-dimensional gauge theories these are magnetic monopole configurations. Explicit instanton calculations have been performed in three-dimensional SU(2) theories with no supersymmetry [18] as well as with $\mathcal{N} = 2$ [19, 20], $\mathcal{N} = 4$ [21, 20] and $\mathcal{N} = 8$ [22] supersymmetry. Discussions of their effects in $\mathcal{N} = 2$ theories with hypermultiplets can be found in [5, 6]. More recently (last week) there has also been a discussion of the effects of instanton-anti-instanton pairs in $\mathcal{N} = 4$ theories [23].

Let us recall a few simple facts. In $\mathcal{N} = 2$ theories, instantons may generate a superpotential. To determine whether or not such a potential does indeed arise in a given case, one must calculate the instanton contribution to the two-fermi correlation function $\langle \mathrm{Tr}\lambda^2 \rangle$, where λ are the vector multiplet fermions. This is non zero only if the instanton has precisely two unlifted fermionic zero modes. Thus we need determine the number of fermionic collective coordinates of a given instanton configuration. These may arise from either vector multiplet or chiral multiplet fermions. The surviving two must be of vector multiplet origin.

The counting of the zero modes follows from the Callias index theorem [24]. A charge k instanton has $2k$ fermionic zero modes donated by the vector multiplet. In contrast, a chiral multiplet of real mass m provides k fermionic zero modes only for $|m| < v$. Otherwise, there are no zero modes.

In the case at hand, the perturbative Coulomb branch is restricted to $0 < v < |m|$. This is precisely the region in which the chiral multiplet fermions provide no zero modes. We therefore find only the $2k$ vector multiplet zero modes and expect a superpotential to be generated by a charge one instanton. Indeed, one can explicitly calculate the contribution to the superpotential from a single instanton,

$$\mathcal{W} = ce^2 \exp(-Y) \sim \left(\frac{c}{e^2}\right) v^3 \sqrt{\frac{|v - m|}{|-v - m|}} \exp\left(\frac{-v}{e^2} + i\sigma\right), \tag{3.3}$$

where we have introduced Y, a holomorphic coordinate on the Coulomb branch, as well as the constant c. This constant has been calculated explicitly in [20] and is non zero.

The second relation in (3.3) is interesting in its own right and we digress somewhat here to explain its significance. It may be determined in two different ways. In the first of these approaches, one calculates the one-loop correction to the complex structure on the Coulomb branch. This determines the corrections to the classical relationship $Y = v/e^2 + i\sigma$. Holomorphicity of the superpotential then ensures that it takes the above form when expressed in terms of the microscopic variables [5]. The $\sim$ sign in (3.3) means "up to two-loop perturbative corrections". In the second, more direct approach, the factor in front of the exponent arises through a calculation of one-loop determinants around the background of the instanton. While in four dimensions, such effects famously cancel in supersymmetric theories [25], the same is not true in three-dimensional theories with less than sixteen supercharges [21]. An explicit calculation of these terms in $\mathcal{N}=2$ theories with matter was performed in [20]. The $\sim$ sign now means "up to two-loops around the background of the instanton". It is intriguing to note that in these theories holomorphicity relates complicated perturbative effects around the background of the instanton to perturbative effects in the vacuum.

In the present situation, the factor in front of the exponent has an important role to play: it ensures the existence of an isolated supersymmetric vacuum. To see this, first recall that supersymmetric vacua are given by,

$$\left|\frac{\partial \mathcal{W}}{\partial Y}\right|^2 = 0 \Longleftrightarrow \mathcal{W} = 0\,,$$

where the "if and only if" refers to the particular superpotential (3.3). We therefore find a supersymmetric vacuum state at $v = m$. This is shown in figure 4. In this vacuum, the chiral multiplet becomes massless and is constrained by a quartic superpotential. It is possible that this point corresponds to an interacting $\mathcal{N}=2$ superconformal theory.

Variations on a theme. One may consider adding various extra matter multiplets to this SU(2) model. For example, the mass M of the second chiral could be kept finite. This then results in disconnected components of the Coulomb branch at $v \geq \max(m, M)$ and $v \leq \min(m, M)$. In the first of these components, no superpotential is generated. In the second, a superpotential similar to (3.3) drives the vacuum to $v = \min(m, M)$ where the superpotential vanishes.

Suppose further that one added a single hypermultiplet (i.e. two chiral multiplets with opposite real mass) in the **2** of the gauge group. We send $M \to \infty$ once more and set the hypermultiplet mass to be equal to $\alpha\, m$ where $0 \leq \alpha \leq 1$. While this hypermultiplet does not alter the perturbative Coulomb branch (it does not generate CS terms), it does affect the superpotential. Indeed, it contributes $2k$ fermionic zero modes to the charge k instanton whenever $v > \alpha|m|$. The superpotential is therefore only generated in the regime $v < \alpha|m|$, and part of the Coulomb branch survives in the quantum theory. Although this appears to violate holomorphy of the superpotential, it was shown in [5, 6] that the Coulomb branch splits about the point $v = \alpha\, m$. This phenomenon is related to the shrinking of the toric fibres in the discussion of the Coulomb branch in the previous section.

There is also a two-dimensional Higgs branch extending from the point $v = \alpha m$. It was argued in [6] that the Higgs and the Coulomb branches merge smoothly at the quantum level. The same behaviour occurs here, the difference being that the Coulomb branch no longer stretches to infinity.

SU(N) theories. We now repeat the analysis of the previous section for SU(N) theories with 2 chiral multiplets transforming in the $\mathbf{N}$ representation. Again, we endow these multiplets with masses $-m$ and $+M$ where $m, M \geq 0$. Unlike the SU(2) model, we will see a somewhat richer behaviour as the mass parameters are varied.

Let us first perform the necessary group theory decomposition that will be needed in both cases. We proceed by first moving onto the Coulomb branch by assigning a vacuum expectation value $\phi = \mathbf{v} \cdot \mathbf{H} = \mathrm{diag}(v_1, \ldots, v_N)$ to the adjoint scalar. Here $\mathbf{v}$ is a $(N-1)$-vector and $\mathbf{H}$ denotes the generators of the Cartan subalgebra. We will find it at times useful to alternate between Cartan-Weyl notation and the more explicit notation in which $\sum_{i=1}^{N} v_i = 0$.

We assume that $\mathbf{v}$ is such that SU(N) breaks to the maximal torus, $\mathrm{U}(1)^{N-1}$. Each chiral multiplet decomposes into N chiral multiplets transforming under the surviving abelian gauge groups, with charges given by $\boldsymbol{\omega}_i \cdot \boldsymbol{\beta}_a C^{ab}$ where $\boldsymbol{\omega}_i$, $i = 1, \ldots, N$ are the fundamental weights of SU(N), $\boldsymbol{\beta}_a$, $a = 1, \ldots, N-1$, are the fundamental roots and C_{ab} is the Cartan matrix. The masses of these chiral multiplets are given by $\boldsymbol{\omega}_i \cdot \mathbf{v} - m = v_i - m$ and $\boldsymbol{\omega}_i \cdot \mathbf{v} + M = v_i + M$. Moreover, in the case of maximal symmetry breaking we may exchange each of the $N-1$ photons for a dual scalar σ_a.

Integrating out the chiral multiplets yields,

$$\kappa_{ab} = \frac{1}{2} \sum_{i=1}^{N} (\boldsymbol{\omega}_i \cdot \boldsymbol{\beta}_a)(\boldsymbol{\omega}_i \cdot \boldsymbol{\beta}_b) \Big(\mathrm{sign}(v_i - m) + \mathrm{sign}(v_i + M) \Big) ,$$

where all a and b indices are to be raised and lowered by the Cartan matrix. Inserting explicit expressions for the weights and roots, we have

$$\begin{aligned} \kappa_{ab} = & \Big[\mathrm{sign}(v_a - m) + \mathrm{sign}(v_a + M) + \mathrm{sign}(v_{a+1} - m) + \mathrm{sign}(v_{a+1} + M) \Big] \delta_{a,b} - \\ & - \Big[\mathrm{sign}(v_a - m) + \mathrm{sign}(v_a + M) \Big] \delta_{a,b+1} - \\ & - \Big[\mathrm{sign}(v_{a+1} - m) + \mathrm{sign}(v_{a+1} + M) \Big] \delta_{b,a+1} . \end{aligned}$$

The existence of a Coulomb branch again requires $\kappa_{ab} = 0$, a requirement which is strongest for the off-diagonal terms, $a = b \pm 1$. These give the condition $-M \leq v_i \leq m$ for each $i = 1, \ldots, N$. Upon fixing the Weyl group, the perturbative Coulomb branch is therefore restricted to

$$m \geq v_1 \geq v_2 \geq \cdots \geq v_N = -\sum_{i=1}^{N-1} v_i \geq -M . \tag{3.4}$$

The perturbative Coulomb branch for SU(3) gauge group in the limit $M \to \infty$ is shown in figure 5.

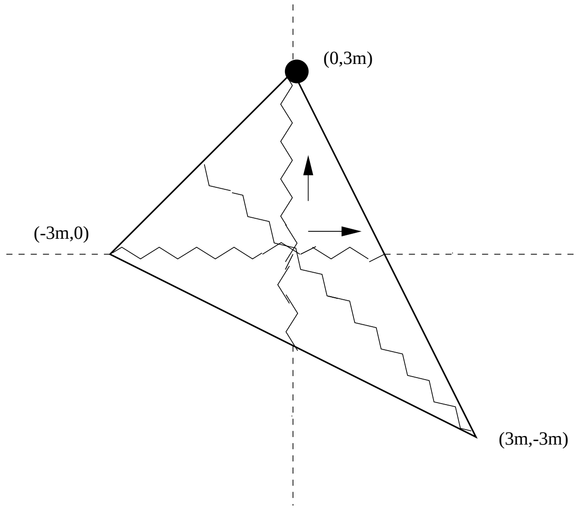

Figure 5: The Coulomb branch of the SU(3) theory with $M \to \infty$. The horizontal and vertical lines are $v_1 - v_2$ and $v_2 - v_3$, respectively. The jagged lines lie at the edge of the Weyl chambers and denote non-abelian symmetry enhancement. Instantons carry the unique supersymmetric vacuum to the spot marked with a dot.

We must now analyse instanton effects in this theory. We again recall some salient facts about monopoles in higher rank gauge groups. The topological charge of such objects is given by a linear combination of co-roots: $\mathbf{g} = \sum_{a=1}^{N-1} k_a \boldsymbol{\beta}_a^\star$ with $k_a \in \mathbb{Z}$. Such a monopole has $2\sum_{a=1}^{N-1} k_a$ fermionic zero modes [26] from vector multiplets. As we are interested in superpotential-generating-instantons, we restrict ourselves to the $N-1$ "fundamental" sectors given by $\mathbf{g} = \boldsymbol{\beta}_a^\star$, each of which have the requisite two fermi zero modes. The story with chiral multiplets is a little more complicated. Having fixed the Weyl symmetry as above, a chiral multiplet of mass real m such that $v_i \geq m \geq v_{i+1}$ donates a single fermi zero mode to only the i-th fundamental monopole. In particular if, as is the case in (3.4), $m \geq v_i$ or $m \leq v_i$ for all i, then it contributes no zero modes. Thus we see that all $(N-1)$ fundamental monopoles can contribute to the superpotential.

Explicit calculations of instanton contributions in SU(N) $\mathcal{N} = 4$ and $\mathcal{N} = 8$ theories, including one-loop factors, were performed in [27] and can be easily generalised to the case at hand. Discussions of the superpotentials generated in $\mathcal{N} = 2$ SU(N) theories are given in [5, 6]. The superpotential is,

$$\begin{aligned} \mathcal{W} &= ce^2 \sum_{a=1}^{N-1} \exp(-Y_a) \qquad (3.5) \\ &\sim \left(\frac{c}{e^2}\right) \sum_{a=1}^{N-1} (\boldsymbol{\beta}_a \cdot \mathbf{v})^3 \sqrt{\frac{|v_a - m|}{|v_{a+1} - m|}} \sqrt{\frac{|v_a + M|}{v_{a+1} - M|}} \exp\left(\boldsymbol{\beta}_a \cdot \mathbf{v}/e^2 + i\sigma_a\right), \end{aligned}$$

where c is a non-zero constant and $\sim$ means once again "up to two-loop effects". The one-loop determinants which make up the prefactor in front of the exponent may be extracted from [5] or [27, 20].

The form of the superpotential once again ensures that the scalar potential vanishes if and only if $\mathcal{W} = 0$. We therefore require $v_a = +m$ or $v_a = -M$ for $a = 1, \ldots, N-1$. Let us choose,

$$\begin{aligned} v_a &= +m\,, \quad a = 1, \ldots, r\,, \\ v_a &= -M\,, \quad a = r+1, \ldots, N-1\,, \\ v_N &= -\sum_{a=1}^{N-1} v_a = -rm + (N-r-1)M\,. \end{aligned} \tag{3.6}$$

Notice that whenever the denominator of (3.5) blows up, the term $(\mathbf{v} \cdot \boldsymbol{\beta}_a)^3$ vanishes faster and the point (3.6) is indeed a zero of $\mathcal{W}$.

We are now in a position to see the fate of this theory. The basic observation is simple: if the putative vacuum (3.6) does not lie in the perturbative Coulomb branch (3.4), supersymmetry is dynamically broken.[4] Let us examine the conditions under which this occurs. The troublesome inequality is that on the right-hand side of (3.4) when combined with the final equation of (3.6). Generically, this latter equation requires $r = N-1$, and it is possible to satisfy the inequality only if

$$M \geq (N-1)m$$

in which case the unique vacuum state lies at $v_a = m$ for $a = 1, \ldots, N-1$ and $v_N = (1-N)m$. For SU(3) gauge group, this vacuum is shown in figure 5. Rather surprisingly, the non-abelian gauge symmetry is partially restored in this vacuum. Specifically, semi-classical analysis suggests that $\mathrm{SU}(N) \to \mathrm{SU}(N-1) \times \mathrm{U}(1)$. However, the dynamics at this point is strongly coupled and the resulting physics is unclear. Certainly the fact that the supersymmetry may be broken by a finite adjustment of the mass parameter M implies that the Witten index must also vanish in the vacuum. It may well be that strong coupling effects conspire to break supersymmetry at this point as well. Moreover, the qualitative low-energy physics of this vacuum is also unclear: is there a mass gap or a non-trivial (super)conformal theory? Clearly it would be interesting to understand these features better.

If $M < (N-1)m$, then the perturbative Coulomb branch contains no zeroes of the superpotential and supersymmetry is generically broken. Similar effects have been seen previously in $\mathcal{N} = 1, 2$ and 3 Chern-Simons theories using brane techniques [28, 29]. For $\mathcal{N} = 1$ theories, the Witten index has also been explicitly computed [30]. The theories studied here appear to be new examples of supersymmetry breaking.

[4] Modulo the usual caveats of singularities of the Kähler potential.

For completeness we note the existence of extra potential supersymmetric vacua (3.6) whenever $r = NM/(m+M) \in \mathbb{Z}$. Such a vacuum has $\mathrm{SU}(r) \times \mathrm{SU}(N-r-1) \times \mathrm{U}(1)$ gauge symmetry. Again, the physics here is poorly understood.

Finally, we briefly mention that one can repeat this analysis for the related $\mathrm{U}(N)$ theory with two chiral multiplets. This has the advantage of being more amenable to brane constructions. Indeed, it is simple to construct the $\mathrm{U}(N)$ version of the above theory on the world-volume of a D3-brane suspended between 5-brane webs [29, 7]. From the field theory side, one finds the same restrictions on the perturbative Coulomb branch and the same superpotential. However, v_N is not constrained to be equal to the sum of the other vacuum expectation values, a relationship that was crucial for the $\mathrm{SU}(N)$ theories exhibiting dynamical supersymmetry breaking. One therefore finds instead a branch of supersymmetric vacua parametrised by v_N, all of which exhibit non-abelian symmetry enhancement. It is simple to show in the brane picture, using the techniques of [31], that a generic configuration of D3-branes will indeed tend towards such a vacuum.[5]

4. Summary

We have shown how to flow from $\mathcal{N}=4$ theories to $\mathcal{N}=2$ theories preserving the mirror map by gauging a suitable combination of the R-symmetries. The resulting theories have Chern-Simons couplings and chiral multiplets. Moreover, we have analysed the dynamics of such theories which display interesting phenomena in their own right. In particular, for a gauge group of rank r, we have:

- The perturbative Coulomb branch is restricted by the CS terms to a region $\Delta \subset \mathbb{R}^r$, which may be compact or non compact depending on the matter content.

- For abelian gauge theories, the dual photons provide a torus $\mathbb{T}^r$ which is fibered over Δ such that certain cycles shrink on the boundaries, resulting in a description of the Coulomb branch as a toric variety.

- For non-abelian gauge groups, a dynamically generated superpotential forces the vacuum to the boundary of Δ where the gauge symmetry is enhanced to a group of rank $r-1$.

- For non-abelian gauge groups of rank $r \geq 2$, certain ranges of the parameters ensure that the vacuum has non-zero energy and the theory exhibits dynamical supersymmetry breaking.

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